

Testing Piecewise Functions

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Abstract

This work explores the query complexity of property testing for general piecewise functions on the real line, in the active and passive property testing settings. The results are proven under an abstract zero-measure crossings condition, which has as special cases piecewise constant functions and piecewise polynomial functions. We find that, in the active testing setting, the query complexity of testing general piecewise functions is independent of the number of pieces. We also identify the optimal dependence on the number of pieces in the query complexity of passive testing in the special case of piecewise constant functions.

Keywords: Property testing; Active testing; Learning theory; Real-valued functions

1. Introduction

Property testing is a well-studied problem, in which an algorithm must decide (with high success probability) from limited observations of some object whether a given property is satisfied, or whether the object is in fact far from all objects with the property. In many property testing problems, it is natural to describe the object as a function f mapping some instance space \mathcal{X} to a value space \mathcal{Y} , and the property as a family of functions \mathcal{F} . The property testing problem is then equivalently stated as determining whether $f \in \mathcal{F}$ or whether f is far from all elements of \mathcal{F} , for a natural notion of distance. In this setting, the observations are simply $(x, f(x))$ points. The setting has been studied in several variations, depending on how the x observation points are selected (e.g., at random or by the algorithm). In particular, this type of property testing problem is closely related to the PAC *learning* model of [1]. One of the main theoretical questions in the study of property testing is how many observations are needed

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by the optimal tester, a quantity known as the *query complexity* of the testing problem. In the above context, this question is most interesting when one can show that the query complexity of the testing problem is significantly smaller than the query complexity of the corresponding PAC learning problem.

Nearly all of the work on the above type of property testing problem has focused on the special case of *binary* functions, where $|\mathcal{Y}| = 2$. In this article, we are interested in initiating the study of more general types of functions, such as real-valued functions. We specifically study one special case of this, wherein $\mathcal{X} = \mathbb{R}$, and the function class \mathcal{F} can be described as a *k-piecewise* function (for a given $k \in \mathbb{N}$), where each piece is a function in a given base class of functions \mathcal{H} . Formally, defining $t_0 = -\infty$ and $t_k = \infty$, for any $t_1, \dots, t_{k-1} \in \mathbb{R}$ with $t_1 \leq \dots \leq t_{k-1}$, and any $h_1, \dots, h_k \in \mathcal{H}$, define (for any $x \in \mathbb{R}$)

$$f(x; \{h_i\}_{i=1}^k, \{t_i\}_{i=1}^{k-1}) = h_i(x) \text{ for the } i \text{ such that } t_{i-1} < x \leq t_i.$$

Then we consider classes $\mathcal{F} = \mathcal{F}_k$ defined as

$$\mathcal{F}_k = \{f(\cdot; \{h_i\}_{i=1}^k, \{t_i\}_{i=1}^{k-1}) : h_1, \dots, h_k \in \mathcal{H}, t_1 \leq \dots \leq t_{k-1}\}.$$

In the results below, we will be particularly interested in the dependence on k in the query complexity of the testing problem.

In this work, our primary running example is the scenario where $\mathcal{Y} = \mathbb{R}$ and \mathcal{H} is the set of *degree-p polynomials*: $\mathcal{H} = \{x \mapsto \sum_{i=0}^p \alpha_i x^i : \alpha_0, \dots, \alpha_p \in \mathbb{R}\}$, so that \mathcal{F}_k is the set of *k-piecewise degree-p polynomials*. A further interesting special case of this is when $p = 0$, in which case \mathcal{F}_k is the set of *k-piecewise constant* functions. Our analysis is abstract, and will also apply to many other interesting function classes \mathcal{H} .

Specifically, for the remainder of this article, we consider \mathcal{Y} as an arbitrary nonempty set (equipped with an appropriate σ -algebra), and \mathcal{H} as an arbitrary nonempty set of measurable functions $\mathcal{X} \rightarrow \mathcal{Y}$ satisfying the following *zero-measure crossings* property:

$$\forall h, h' \in \mathcal{H}, \quad h \neq h' \implies \lambda(\{x : h(x) = h'(x)\}) = 0, \quad (1)$$

where λ denotes the Lebesgue measure. It is well known that this property is satisfied by polynomial functions: for any two polynomial functions h, h' of degree p , if they agree on a nonzero measure set of points, then we can find in that set distinct points x_1, \dots, x_{p+1} on which they agree, but since the values on any $p + 1$ distinct points uniquely determine the polynomial function, it must be that $h = h'$. Thus, the analysis below indeed applies to piecewise polynomial functions as a special case. The zero-measure crossings property is also satisfied by many other interesting function classes, such as shifted sine functions $\{x \mapsto \sin(x + t) : t \in \mathbb{R}\}$ or normal pdfs $\{x \mapsto c \cdot e^{-(x-t)^2/2} : t \in \mathbb{R}\}$.

The testing problem is defined as follows. Fix a value $\epsilon \in (0, 1)$ and a probability measure \mathcal{P} over \mathcal{X} , and for any measurable $f, g : \mathcal{X} \rightarrow \mathcal{Y}$, define $\rho(f, g) = \mathcal{P}(x : f(x) \neq g(x))$, the $L_0(\mathcal{P})$ (pseudo)distance between f and g . Further define $\rho(f, \mathcal{F}) = \inf_{g \in \mathcal{F}} \rho(f, g)$, the distance of f from a set of functions

\mathcal{F} . In the active testing protocol, the algorithm samples a number s of iid unlabeled examples from \mathcal{P} , and then interactively queries for the f values at points of its choosing from among these: that is, it chooses one of the s examples x , queries for its value $f(x)$, then selects another of the s examples x' , queries for its value $f(x')$, and so on, until it eventually halts and produces a decision of either ACCEPT or REJECT. The following definition is taken from [2].

Definition 1. *An s -sample q -query ϵ -tester for \mathcal{F} under the distribution \mathcal{P} is a randomized algorithm A that draws a sample S of size s iid from \mathcal{P} , sequentially queries for the value of f on q points of S , and satisfies the following properties (regardless of the identity of f):*

- if $f \in \mathcal{F}$, it decides ACCEPT with probability at least $\frac{2}{3}$.
- if $\rho(f, \mathcal{F}) \geq \epsilon$, it decides REJECT with probability at least $\frac{2}{3}$.

We will be interested in identifying values of q for which there exist s -sample q -query ϵ -testers for \mathcal{F} , known as the *query complexity*; we are particularly interested in this when s is polynomial in k , $1/\epsilon$, and the complexity of \mathcal{H} (defined below). In the special case that $q = s$, so that the algorithm queries the values at *all* of the points in S , the algorithm is called a *passive tester*, whereas in the general case of $q \leq s$ it is referred to as an *active tester* (in analogy to the setting of *active learning* studied in the machine learning literature).

Remark on general distributions \mathcal{P} : For simplicity, we will only consider $\mathcal{P} = \text{Uniform}(0, 1)$ in this work. We note, however, that all of our results easily generalize to all distributions \mathcal{P} over \mathbb{R} absolutely continuous with respect to Lebesgue measure. Specifically, [2] discuss a simple trick which produces this generalization, by using the empirical distribution to effectively rescale the real line so that the distribution appears approximately uniform. One can show that this trick is also applicable in the present more-general context as well. The rescaling effectively changes the function class \mathcal{H} , but its graph dimension (defined below) remains unchanged, and the zero-measure crossings property is preserved due to \mathcal{P} being absolutely continuous. In the special case of testing piecewise *constant* functions, even the restriction to absolutely continuous distributions can be removed, as then the zero-measure crossings property always holds. The interested reader is referred to [2] for the details of this rescaling trick.

1.1. Related Work: Testing Unions of Intervals

The work of [2] explored the query complexity of testing in both the active and passive models, for a variety of function classes, but all under the restriction $\mathcal{Y} = \{0, 1\}$. Of the results of [2], the most relevant to the present work are results on the query complexity of testing *unions of intervals*: $x \mapsto \mathbb{I}[x \in \bigcup_{i=1}^n [t_{2i-1}, t_{2i}]]$, for a fixed $n \in \mathbb{N}$, defined for all nondecreasing sequences $t_1, \dots, t_{2n} \in \mathbb{R} \cup \{\pm\infty\}$. They specifically find that the query complexity of testing unions of intervals is $\Theta(1)$ in the active testing setting, and $\Theta(\sqrt{n})$ in the

passive testing setting (ignoring the dependence on ϵ); these results strengthened an earlier result of Kearns and Ron [3]. Note that unions of intervals are a special case of piecewise constant functions, and indeed the techniques we employ in constructing the tester below closely parallel the analysis of unions of intervals by [2]. However, to extend that approach to general piecewise functions — even piecewise *constant* functions — requires careful consideration about what the appropriate generalization of the technique should be. In particular, the original proof involved a *self-correction* step, wherein f is replaced by a smoothed function, which is then rounded again to a binary-valued function. This kind of smoothing and rounding interpretation no longer makes sense for general \mathcal{Y} -valued functions. We are therefore required to re-interpret these steps in the more general setting, where we find that they can be reformulated as *voting* on the function value at each point (rather than rounding a smoothed version of f). In the end, the active tester below for general \mathcal{F}_k functions has significant differences compared to the original tester for unions of intervals from [2]. Nevertheless, we do recover the dependences on k established by [2] for active and passive testing of unions of intervals, as special cases of our general results on testing piecewise functions. These considerations also lead us to an appropriate generalization beyond binary-valued functions for the notion of *noise sensitivity*, a quantity commonly appearing in the property testing literature [2].

1.2. The Graph Dimension

For any set \mathcal{Z} , any collection \mathcal{C} of subsets of \mathcal{Z} , and any $m \in \mathbb{N} \cup \{0\}$, following [4], we say \mathcal{C} *shatters* a set $\{z_1, \dots, z_m\} \subseteq \mathcal{Z}$ if

$$|\{C \cap \{z_1, \dots, z_m\} : C \in \mathcal{C}\}| = 2^m.$$

The VC dimension of \mathcal{C} is then defined as the largest integer m for which $\exists \{z_1, \dots, z_m\} \subseteq \mathcal{Z}$ shattered by \mathcal{C} , or as infinity if no such largest m exists.

The VC dimension is an important quantity in characterizing the optimal sample complexity of statistical learning for binary classification. It is also useful in our present context, for the purpose of defining a related complexity measure for general functions. Specifically, the *graph dimension* of \mathcal{H} , denoted d below, is defined as the VC dimension of the collection $\{(x, h(x)) : x \in \mathcal{X}\} : h \in \mathcal{H}$ (where $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ in this context). For the remainder of this article, we restrict to the case $0 < d < \infty$, but otherwise we can consider \mathcal{H} as a completely *arbitrary* set of functions (subject to (1)).

To continue the examples from above, we note that in the case of $|\mathcal{Y}| \geq 2$ and \mathcal{H} as the set of constant functions (so that \mathcal{F}_k is the k -piecewise constant functions) one can easily show $d = 1$. Moreover, in the case of $\mathcal{Y} = \mathbb{R}$ and \mathcal{H} as the set of degree- p real-valued polynomial functions (so that \mathcal{F}_k is the k -piecewise degree- p polynomial functions), it follows from basic algebra that $d = p + 1$, since any polynomial f is uniquely determined by any $p + 1$ distinct $(x, f(x))$ pairs (so that $d \leq p + 1$), and any $p + 1$ pairs (x, y) (with distinct x components) can be fit by a polynomial (so that $d \geq p + 1$, by choosing $p + 1$

distinct x points, each with two corresponding y values, and all 2^{p+1} choices of which y value to use for each point can be fit by a degree- p polynomial).

The results below will be expressed in terms of k , ϵ , and d . The dependence of the query complexity on each of these is an important topic to consider. However, in the present work, we primarily focus on identifying the optimal dependence on k . In particular, in all asymptotic statements, we will consider \mathcal{H} and ϵ as *fixed*, so that k is the only asymptotic variable: for instance, we may write $kd/\epsilon = O(k)$. The optimal dependence on ϵ and the complexity of \mathcal{H} is a problem we leave for future work. In contexts where we wish to describe the dependence on all of k , ϵ , and d , we write $M \lesssim \phi(k, d, \epsilon)$, for any function ϕ and value M , to mean that there exists a numerical constant c (independent of the choices of k , ϵ , and d) such that $M \leq c\phi(k, d, \epsilon)$.

1.3. Main Results

We are now ready to state our main results. We present their proofs in the next section. The first result is for active testing.

Theorem 1. *For any $\epsilon \in (0, 1/2)$, there exists an s -sample q -query ϵ -tester for \mathcal{F}_k under the distribution \mathcal{P} , with $s \lesssim \frac{dk}{\epsilon^8} \ln(\frac{1}{\epsilon}) = \text{poly}(k, d, 1/\epsilon)$ and $q \lesssim \frac{d}{\epsilon^{12}} \ln(\frac{1}{\epsilon}) = O(1)$.*

In particular, this immediately implies that the optimal query complexity of active testing is $\Theta(1)$.

Our second result is for *passive* testing, and applies specifically to the special case of piecewise *constant* functions (for any \mathcal{Y} space).

Theorem 2. *If \mathcal{H} is the set of constant functions, then for any $\epsilon \in (0, 1/2)$, there exists a q -sample q -query ϵ -tester for \mathcal{F}_k under the distribution \mathcal{P} , with $q \lesssim \frac{\sqrt{k}}{\epsilon^7} = O(\sqrt{k})$.*

Moreover, in this case, if $\epsilon \in (0, 1/8)$, every q -sample q -query ϵ -tester for \mathcal{F}_k under the distribution \mathcal{P} has $q = \Omega(\sqrt{k})$.

In particular, this implies that the optimal query complexity of passive testing is $\Theta(\sqrt{k})$ for k -piecewise constant functions. It is also straightforward to extend this lower bound to k -piecewise degree- p polynomial functions. However, our results below do not imply an $O(\sqrt{k})$ upper bound for passive testing of this larger function class, and as such identifying the optimal query complexity of passive testing for piecewise degree- p polynomials (and for general classes \mathcal{H} satisfying (1)) remains an interesting open problem.

2. Proofs

This section is devoted to presenting the proofs of the above theorems.

2.1. A Generalization of Noise Sensitivity

Here we develop a generalization of the definition of the *noise sensitivity* used by [2] in their analysis of testing unions of intervals. For any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, define $\mathcal{H}_{(x,y)} = \{h \in \mathcal{H} : h(x) = y\}$. Let $x \sim \text{Uniform}(0,1)$, and conditioned on x , let $x' \sim \text{Uniform}(x - \delta, x + \delta)$. Define the *instantaneous noise sensitivity* at x as¹

$$\text{NS}_\delta(f, x) = \inf_{h \in \mathcal{H}_{(x, f(x))}} \mathbb{P}(h(x') \neq f(x') | x),$$

or in the event that $\mathcal{H}_{(x, f(x))}$ is empty, define $\text{NS}_\delta(f, x) = 1$. Then define the *noise sensitivity* as

$$\text{NS}_\delta(f) = \mathbb{E}[\text{NS}_\delta(f, x)] = \int_0^1 \text{NS}_\delta(f, z) dz.$$

We have the following two key lemmas on this definition of noise sensitivity. Their statements and proofs directly parallel the analysis of unions of intervals by [2], but with a few important changes (particularly in the proof of Lemma 2) to generalize the arguments to general piecewise functions.

Lemma 1. *For any $\delta > 0$, $\forall f \in \mathcal{F}_k$, $\text{NS}_\delta(f) \leq (k-1)\frac{\delta}{2}$.*

Proof. For $f \in \mathcal{F}_k$, let $h_1, \dots, h_k \in \mathcal{H}$ and $t_1, \dots, t_{k-1} \in \mathbb{R}$ be such that $f(\cdot) = f(\cdot; \{h_i\}_{i=1}^k, \{t_i\}_{i=1}^{k-1})$. Thus, for any $i \leq k$ and $x \in (t_{i-1}, t_i]$, and any $y \in \mathbb{R}$, we have $h_i(x) = f(x)$, and we would have $h_i(y) \neq f(y)$ only if x and y are separated by one of the boundaries t_{i-1} or t_i ; in particular, $x \leq t_i \leq y$ or $y \leq t_{i-1} \leq x$.

Let $x \sim \text{Uniform}(0,1)$ and (conditioned on x) $y \sim \text{Uniform}(x - \delta, x + \delta)$. Denoting by $i(x)$ the i with $t_{i-1} < x \leq t_i$, we have

$$\text{NS}_\delta(f, x) \leq \mathbb{P}(h_{i(x)}(y) \neq f(y) | x) \leq \mathbb{P}(x \leq t_{i(x)} \leq y | x) + \mathbb{P}(y \leq t_{i(x)-1} \leq x | x),$$

so that

$$\begin{aligned} \text{NS}_\delta(f) &\leq \mathbb{P}(x \leq t_{i(x)} \leq y) + \mathbb{P}(y \leq t_{i(x)-1} \leq x) \\ &\leq \sum_{i=1}^{k-1} (\mathbb{P}(x \leq t_i \leq y) + \mathbb{P}(y \leq t_i \leq x)), \end{aligned}$$

where the last inequality uses the fact that $\mathbb{P}(t_k \leq y) = \mathbb{P}(y \leq t_0) = 0$.

For any fixed $t \in \mathbb{R}$,

$$\mathbb{P}(x \leq t \leq y) \leq \int_0^\delta \mathbb{P}_{y' \sim \text{Uniform}(t-z-\delta, t-z+\delta)}[y' \geq t] dz = \int_0^\delta \frac{\delta - z}{2\delta} dz = \frac{\delta}{4},$$

¹The original definition of [2] essentially defined $\text{NS}_\delta(f, x) = \mathbb{P}(f(x') \neq f(x) | x)$. The involvement of \mathcal{H} in our generalization of the definition will be crucial to the results below.

noting that, if t is outside $[\delta, 1]$, then the probability can only become smaller. Similarly, any $t \in \mathbb{R}$ has $\mathbb{P}(y \leq t \leq x) \leq \frac{\delta}{4}$, again noting that the probability only becomes smaller if t is outside $[0, 1 - \delta]$.

Plugging these inequalities into the above bound on $\mathbb{NS}_\delta(f)$ yields $\mathbb{NS}_\delta(f) \leq (k-1)\frac{\delta}{2}$, as claimed. \square

Lemma 2. Fix any $\epsilon \in (0, 1/2)$ and let $\delta = \frac{\epsilon^2}{32k}$. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be any function with $\mathbb{NS}_\delta(f) \leq (k-1)\frac{\delta}{2}(1 + \frac{\epsilon}{4})$. Then $\rho(f, \mathcal{F}_k) < \epsilon$.

Proof. Let $k' = \lfloor 1 + (k-1)(1 + \frac{\epsilon}{2}) \rfloor$. We first argue that f is $\frac{\epsilon}{2}$ -close to a function in $\mathcal{F}_{k'}$, and then we argue that every function in $\mathcal{F}_{k'}$ is $\frac{\epsilon}{2}$ -close to \mathcal{F}_k .

For each $h \in \mathcal{H}$, consider the function $f_\delta^h : [0, 1] \rightarrow [0, 1]$ defined by

$$f_\delta^h(x) = \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} \mathbb{I}[f(t) = h(t)] dt.$$

The function f_δ^h is the convolution of $t \mapsto \mathbb{I}[f(t) = h(t)]$ and the uniform kernel $\phi : \mathbb{R} \rightarrow [0, 1]$ defined by $\phi(x) = \frac{1}{2\delta} \mathbb{I}[|x| \leq \delta]$. Note that, since any distinct $h, h' \in \mathcal{H}$ have $\int_0^1 \mathbb{I}[h(t) = h'(t)] dt = 0$ (by the zero-measure crossings assumption (1)), the sum (over $h \in \mathcal{H}$) of all $f_\delta^h(x)$ values is at most 1. In particular, at most one $h \in \mathcal{H}$ has $f_\delta^h(x) > 1/2$ for any x .

Fix $\tau = \frac{4}{\epsilon} \mathbb{NS}_\delta(f)$. Since $\mathbb{NS}_\delta(f) \leq (k-1)\frac{\delta}{2}(1 + \frac{\epsilon}{4}) < \frac{\epsilon^2}{32}$, we have $\tau < 1/8$. For each x , let $h_x = \operatorname{argmax}_{h \in \mathcal{H}} f_\delta^h(x)$ (breaking ties arbitrarily); since the sum (over $h \in \mathcal{H}$) of $f_\delta^h(x)$ values is finite (bounded by 1), it follows that the value $\sup_{h \in \mathcal{H}} f_\delta^h(x)$ is actually realized by some $f_\delta^h(x)$ with $h \in \mathcal{H}$, so that h_x is well-defined. Define a function $g^* : [0, 1] \rightarrow \mathcal{Y} \cup \{*\}$ by $g^*(x) = h_x(x)$ if $f_\delta^{h_x}(x) \geq 1 - \tau$, and $g^*(x) = *$ otherwise. Next, define a function $g : \mathbb{R} \rightarrow \mathcal{Y}$ by setting, for any $x \in [0, 1]$, $g(x) = h_z(x)$ where z is the largest value in $[0, x]$ for which $g^*(z) \neq *$, and let $g_x = h_z$; if no such z exists, take z minimal in $[x, 1]$ with $g^*(z) \neq *$ instead; if that also does not exist, we can define $g(x)$ and g_x arbitrarily, as this case will not come up in our present context. As we discuss below, $f_\delta^{h_x}(x) = \sup_{h \in \mathcal{H}} f_\delta^h(x)$ is continuous in x , which entails that at least one of these two possible z values will exist if g^* is not everywhere equal $*$ in $[0, 1]$. For completeness, also define, for any $x < 0$, $g(x) = g_0(x)$, and for any $x > 1$, $g(x) = g_1(x)$.

Now note that

$$\begin{aligned} \rho(f, g) &= \mathcal{P}(x : f(x) \neq g(x)) \leq \mathcal{P}(x : g^*(x) = *) + \mathcal{P}(x : * \neq g^*(x) \neq f(x)) \\ &= \mathcal{P}\left(x : \sup_{h \in \mathcal{H}} f_\delta^h(x) < 1 - \tau\right) \\ &\quad + \mathcal{P}\left(x : \mathcal{H} \setminus \mathcal{H}_{(x, f(x))} \neq \emptyset, \sup_{h \in \mathcal{H} \setminus \mathcal{H}_{(x, f(x))}} f_\delta^h(x) \geq 1 - \tau\right). \end{aligned} \quad (2)$$

Because $\tau < 1/2$, at most one h can have $f_\delta^h(x) \geq 1 - \tau$ (as discussed above), so that if the event $\sup_{h \in \mathcal{H}} f_\delta^h(x) < 1 - \tau$ holds or the event $\mathcal{H} \setminus \mathcal{H}_{(x, f(x))} \neq \emptyset$

and $\sup_{h \in \mathcal{H} \setminus \mathcal{H}_{(x, f(x))}} f_\delta^h(x) \geq 1 - \tau$ holds, then either way we would have $\sup_{h \in \mathcal{H}_{(x, f(x))}} f_\delta^h(s) < 1 - \tau$ (or $\mathcal{H}_{(x, f(x))} = \emptyset$); thus, since the two events implying this are disjoint, the sum of probabilities in (2) is at most

$$\mathcal{P}\left(x : \mathcal{H}_{(x, f(x))} = \emptyset \text{ or } \sup_{h \in \mathcal{H}_{(x, f(x))}} f_\delta^h(x) < 1 - \tau\right).$$

Now observe that $\text{NS}_\delta(f, x) = 1 - \sup_{h \in \mathcal{H}_{(x, f(x))}} f_\delta^h(x)$ if $\mathcal{H}_{(x, f(x))} \neq \emptyset$, and $\text{NS}_\delta(f, x) = 1$ if $\mathcal{H}_{(x, f(x))} = \emptyset$. Together with Markov's inequality, this implies that

$$\begin{aligned} \mathcal{P}\left(x : \mathcal{H}_{(x, f(x))} = \emptyset \text{ or } \sup_{h \in \mathcal{H}_{(x, f(x))}} f_\delta^h(x) < 1 - \tau\right) \\ = \mathcal{P}(x : \text{NS}_\delta(f, x) > \tau) < \frac{\text{NS}_\delta(f)}{\tau} = \frac{\epsilon}{4}. \end{aligned}$$

Thus, we have established that $\rho(f, g) \leq \frac{\epsilon}{4}$.

Next we show that $g \in \mathcal{F}_{m+1}$ for some nonnegative integer $m \leq (k-1)(1 + \frac{\epsilon}{2})$. Since each f_δ^h is the convolution of $\mathbb{I}[f(\cdot) = h(\cdot)]$ with a uniform kernel of width 2δ , it is $\frac{1}{2\delta}$ -Lipschitz smooth. Also recall that $\tau < 1/2$, and the sum of all $f_\delta^h(x)$ values for a given x is at most 1. Thus, if we consider any two points $x, z \in [0, 1]$ with $g^*(x) \neq *$, $g^*(z) \neq *$, $x < z$, and $h_x \neq h_z$, then it must be that $|x - z| \geq 2\delta(1 - 2\tau)$, and that there is at least one point $t \in (x, z)$ with $\sup_{h \in \mathcal{H}} f_\delta^h(t) = 1/2$. Since each f_δ^h is $\frac{1}{2\delta}$ -Lipschitz, so is $\sup_{h \in \mathcal{H}} f_\delta^h$, so that we have

$$\int_{t-2\delta(\frac{1}{2}-\tau)}^{t+2\delta(\frac{1}{2}-\tau)} \sup_{h \in \mathcal{H}} f_\delta^h(s) ds \leq 2 \int_0^{2\delta(\frac{1}{2}-\tau)} \left(\frac{1}{2} + \frac{s}{2\delta}\right) ds = 2\delta \left(\frac{1}{2} - \tau\right) \left(\frac{3}{2} - \tau\right).$$

Therefore,

$$\begin{aligned} \int_x^z \text{NS}_\delta(f, s) ds &\geq \int_x^z \left(1 - \sup_{h \in \mathcal{H}} f_\delta^h(s)\right) ds \\ &\geq (z - x) - 2\delta \left(\frac{1}{2} - \tau\right) \left(\frac{3}{2} - \tau\right) \geq 2\delta(1 - 2\tau) - 2\delta \left(\frac{1}{2} - \tau\right) \left(\frac{3}{2} - \tau\right) \\ &= 2\delta \left(\frac{1}{2} - \tau\right) \left(\frac{1}{2} + \tau\right) = 2\delta \left(\frac{1}{4} - \tau^2\right). \end{aligned}$$

Since any x with $g^*(x) \neq *$ has $g(x) = g^*(x)$, and since g_t is extrapolated from the left in $*$ regions of g^* (aside from the case of an interval of $*$ values including 0, where it is extrapolated from the right), for every point $x > 0$ for which there exist arbitrarily close points y having $g_y \neq g_x$, we must have that $g^*(x) \neq *$, and that there is a point $z < x$ such that $g^*(z) \neq *$ and such that every $t \in (z, x)$ has $g_t = g_z \neq g_x$. Combined with the above, we have that

$\int_z^x \text{NS}_\delta(f, s) ds \geq 2\delta(\frac{1}{4} - \tau^2)$. Altogether, if g has m such “transition” points, then

$$\text{NS}_\delta(f) = \int_0^1 \text{NS}_\delta(f, s) ds \geq m2\delta \left(\frac{1}{4} - \tau^2 \right).$$

By assumption, $\text{NS}_\delta(f) \leq (k-1)\frac{\delta}{2}(1 + \frac{\epsilon}{4})$. Therefore, we must have

$$m \leq \frac{(k-1)\delta(1 + \frac{\epsilon}{4})}{4\delta(\frac{1}{4} - \tau^2)} \leq (k-1) \frac{1 + \frac{\epsilon}{4}}{1 - 4\tau^2} \leq (k-1) \frac{1 + \frac{\epsilon}{4}}{(1 - 2\tau)^2} \leq (k-1) \left(1 + \frac{\epsilon}{2} \right),$$

since $\tau < 1/8$. In particular, this means $g \in \mathcal{F}_{m+1}$ for an $m \leq (k-1)(1 + \frac{\epsilon}{2})$, as claimed.

As a second step in the proof, we show that for any nonnegative integer $m \leq (k-1)(1 + \frac{\epsilon}{2})$, any function $g' \in \mathcal{F}_{m+1}$ is $\frac{\epsilon}{2}$ -close to a function in \mathcal{F}_k . Let $t_1, \dots, t_m \in \mathbb{R}$ with $t_1 \leq \dots \leq t_m$, and $h_1, \dots, h_{m+1} \in \mathcal{H}$, be such that $g'(\cdot) = f(\cdot; \{h_i\}_{i=1}^{m+1}, \{t_i\}_{i=1}^m)$. For each $i \in \{1, \dots, m+1\}$, let $\ell_i = \mathcal{P}((t_{i-1}, t_i])$ denote the probability mass in the i^{th} region. In particular, $\ell_1 + \dots + \ell_{m+1} = 1$, so there must be a set $S \subseteq \{1, \dots, m+1\}$ with $|S| = (m+1) - k \leq (k-1)\frac{\epsilon}{2}$ such that

$$\sum_{i \in S} \ell_i \leq \frac{(m+1) - k}{(m+1)} \leq \frac{(k-1)\epsilon/2}{1 + (k-1)(1 + \epsilon/2)} < \frac{\epsilon}{2}.$$

Define a function $f' : \mathcal{X} \rightarrow \mathcal{Y}$ such that, for each $i \in \{1, \dots, m+1\}$ and $x \in (t_{i-1}, t_i]$, we set $f'(x) = h_j(x)$ for the $j \in \{1, \dots, m+1\} \setminus S$ of smallest $|i - j|$ (breaking ties to favor smaller j). The function f' is then contained in \mathcal{F}_k , and has $f'(x) = g'(x)$ for every $x \in (t_{i-1}, t_i]$ with $i \notin S$, and hence $\rho(g', f') < \frac{\epsilon}{2}$. This completes the proof, since taking $g' = g$ yields $\rho(f, f') \leq \rho(f, g) + \rho(g, f') < \epsilon$. \square

2.2. Active Testing

We can use the above lemmas to construct an active tester for \mathcal{F}_k as follows. Fix any $\epsilon \in (0, 1/2)$. Let $m = \lceil \frac{c}{\epsilon^6} \ln(24) \rceil$, $\ell = \lceil \frac{c'd}{\epsilon^6} \ln\left(\frac{12m \cdot 8ec'}{\epsilon^6}\right) \rceil$, $\delta = \frac{\epsilon^2}{32k}$, and $s = m + \lceil \max\{\frac{2\ell}{\delta}, \frac{8}{\delta} \ln(12m)\} \rceil$, for appropriate choices of numerical constants $c, c' \geq 1$ from the analysis below. Sample s points x''_1, \dots, x''_s independent Uniform(0, 1). Define $x_i = x''_i$ for each $i \leq m$. For each $i \leq m$, denoting $t_{i0} = m$, for each $j \in \{1, \dots, \ell\}$, let $t_{ij} = \min\{t \in \{t_{i(j-1)} + 1, \dots, s\} : x''_t \in (x_i - \delta, x_i + \delta)\}$ if such a value exists (if it does not exist, the tester may return any response, as this is a failure case), and define $x'_{ij} = x''_{t_{ij}}$. Thus, the random variables x_1, \dots, x_m are iid Uniform(0, 1) and, given x_i , the random variables $x'_{i1}, \dots, x'_{i\ell}$ are conditionally iid Uniform($(x_i - \delta, x_i + \delta) \cap [0, 1]$) (given x_i and the event that they exist). The tester requests the f values for all $m(\ell + 1)$ of these points $x_i, x'_{ij}, i \in \{1, \dots, m\}, j \in \{1, \dots, \ell\}$. It then calculates, for each $i \leq m$,

$$\hat{\text{NS}}_\delta(f, x_i) = \min_{h \in \mathcal{H}_{(x_i, f(x_i))}} \frac{1}{\ell} \sum_{j=1}^{\ell} \mathbb{I}[h(x'_{ij}) \neq f(x'_{ij})],$$

or $\hat{\text{NS}}_\delta(f, x_i) = 1$ in the event that $\mathcal{H}_{(x_i, f(x_i))}$ is empty. Then define

$$\hat{\text{NS}}_\delta(f) = \frac{1}{m} \sum_{i=1}^m \hat{\text{NS}}_\delta(f, x_i).$$

Lemma 3. *For appropriate choices of numerical constants c, c' , for any measurable function $f : \mathcal{X} \rightarrow \mathcal{Y}$, with probability at least $2/3$, all of the above x'_{ij} points exist, and $|\hat{\text{NS}}_\delta(f) - \text{NS}_\delta(f)| \leq 2\delta + \epsilon^3/2^{10}$.*

Proof. For each $i \leq m$, any $t \in \{m+1, \dots, s\}$ has conditional probability (given x_i) at least δ of having $x''_t \in (x_i - \delta, x_i + \delta)$. Therefore, by a Chernoff bound (applied under the conditional distribution given x_i) and the law of total probability, with probability at least $1 - \exp\{-\delta(s-m)/8\}$, the number of $t \in \{m+1, \dots, s\}$ with $x''_t \in (x_i - \delta, x_i + \delta)$ is at least $(1/2)\delta(s-m) \geq \ell$. By the union bound, this holds simultaneously for all $i \leq m$ with probability at least $1 - m \exp\{-\delta(s-m)/8\} \geq 11/12$, and on this event all of the x'_{ij} points exist.

The VC dimension of the collection of sets $\{(x, h(x)) : x \in \mathcal{X}\} : h \in \mathcal{H}$ is d (by definition of d). Therefore, applying standard VC uniform convergence bounds [4] to obtain a uniform concentration inequality for the frequency of $(x'_{ij}, f(x'_{ij})) \in \{(x, h(x)) : x \in \mathcal{X}\}$, holding for all $h \in \mathcal{H}$, we obtain that, for an appropriate choice of universal constant c' (e.g., $c' = 2^{28}$ would suffice), for each $i \leq m$, with probability at least $1 - 1/(12m)$, if the points x'_{ij} exist, then every $h \in \mathcal{H}$ has

$$\left| \frac{1}{\ell} \sum_{j=1}^{\ell} \mathbb{I}[h(x'_{ij}) \neq f(x'_{ij})] - \mathbb{P}(h(x'_{i1}) \neq f(x'_{i1}) | x_i) \right| \leq \frac{\epsilon^3}{2^{12}}.$$

The union bound implies this is true simultaneously for all $i \leq m$ with probability at least $11/12$. Furthermore, for any $x_i \in (\delta, 1 - \delta)$, the conditional distribution of x'_{i1} given x_i is $\text{Uniform}(x_i - \delta, x_i + \delta)$, so that $\text{NS}_\delta(f, x_i) = \inf_{h \in \mathcal{H}_{(x_i, f(x_i))}} \mathbb{P}(h(x'_{i1}) \neq f(x'_{i1}) | x_i)$ in the case $\mathcal{H}_{(x_i, f(x_i))} \neq \emptyset$. Thus, on the above events, for each $i \leq m$ with $x_i \in (\delta, 1 - \delta)$ and $\mathcal{H}_{(x_i, f(x_i))} \neq \emptyset$,

$$\hat{\text{NS}}_\delta(f, x_i) \leq \inf_{h \in \mathcal{H}_{(x_i, f(x_i))}} \mathbb{P}(h(x'_{i1}) \neq f(x'_{i1}) | x_i) + \frac{\epsilon^3}{2^{12}} = \text{NS}_\delta(f, x_i) + \frac{\epsilon^3}{2^{12}},$$

and similarly

$$\hat{\text{NS}}_\delta(f, x_i) \geq \inf_{h \in \mathcal{H}_{(x_i, f(x_i))}} \mathbb{P}(h(x'_{i1}) \neq f(x'_{i1}) | x_i) - \frac{\epsilon^3}{2^{12}} = \text{NS}_\delta(f, x_i) - \frac{\epsilon^3}{2^{12}},$$

so that together we have

$$\left| \hat{\text{NS}}_\delta(f, x_i) - \text{NS}_\delta(f, x_i) \right| \leq \frac{\epsilon^3}{2^{12}}.$$

This is trivially also satisfied in the case $\mathcal{H}_{(x_i, f(x_i))} = \emptyset$. Furthermore, by Hoeffding's inequality, for an appropriate choice of numerical constant c (e.g., any $c \geq 2^{21}$ would suffice), with probability at least $11/12$,

$$\frac{1}{m} \sum_{i=1}^m \mathbb{I}[x_i \notin (\delta, 1 - \delta)] \leq 2\delta + \frac{\epsilon^3}{2^{11}}.$$

Additionally, another application of Hoeffding's inequality implies that, for an appropriate choice of numerical constant c (e.g., any $c \geq 2^{23}$ would suffice), with probability at least $11/12$,

$$\left| \frac{1}{m} \sum_{i=1}^m \text{NS}_\delta(f, x_i) - \text{NS}_\delta(f) \right| \leq \frac{\epsilon^3}{2^{12}}.$$

The union bound implies that all four of the above events hold simultaneously with probability at least $2/3$. Moreover, when these hold, the above inequalities together imply that

$$\begin{aligned} & |\hat{\text{NS}}_\delta(f) - \text{NS}_\delta(f)| \\ & \leq \left| \frac{1}{m} \sum_{i=1}^m \text{NS}_\delta(f, x_i) - \text{NS}_\delta(f) \right| + \left| \frac{1}{m} \sum_{i=1}^m \hat{\text{NS}}_\delta(f, x_i) - \frac{1}{m} \sum_{i=1}^m \text{NS}_\delta(f, x_i) \right| \\ & \leq \frac{\epsilon^3}{2^{12}} + \frac{1}{m} \sum_{i=1}^m \left| \hat{\text{NS}}_\delta(f, x_i) - \text{NS}_\delta(f, x_i) \right| \\ & \leq \frac{\epsilon^3}{2^{11}} + \frac{1}{m} \sum_{i=1}^m \mathbb{I}[x_i \notin (\delta, 1 - \delta)] \leq 2\delta + \frac{\epsilon^3}{2^{10}}. \end{aligned}$$

□

We are now ready to finish describing the tester and prove its correctness.

Theorem 3. *If $k \geq 80/\epsilon$, then the procedure that outputs ACCEPT if $\hat{\text{NS}}_\delta(f) \leq (k-1)\frac{\delta}{2}(1 + \frac{\epsilon}{8})$, and otherwise outputs REJECT, is an s -sample q -query ϵ -tester for the class \mathcal{F}_k of k -piecewise \mathcal{H} functions under the distribution \mathcal{P} , for s as defined above, and for $q = m(\ell + 1)$ (where m and ℓ are as defined above).*

Proof. First note that, since $k \geq 80/\epsilon$, it holds that

$$\begin{aligned} 2\delta + \frac{\epsilon^3}{2^{10}} &= (k-1)\frac{\delta}{2} \left(\frac{4}{k-1} + \frac{k}{k-1} \frac{\epsilon}{16} \right) \\ &\leq (k-1)\frac{\delta}{2} \frac{\epsilon}{4} \frac{k}{k-1} \left(\frac{1}{5} + \frac{1}{4} \right) \leq (k-1)\frac{\delta}{2} \frac{\epsilon}{8}. \end{aligned}$$

If $f \in \mathcal{F}_k$, then Lemma 1 implies it has $\text{NS}_\delta(f) \leq (k-1)\frac{\delta}{2}$, so that Lemma 3 implies that with probability at least $2/3$, $\hat{\text{NS}}_\delta(f) \leq (k-1)\frac{\delta}{2} + 2\delta + \frac{\epsilon^3}{2^{10}} \leq (k-1)\frac{\delta}{2} (1 + \frac{\epsilon}{8})$, and hence the tester will output ACCEPT.

On the other hand, if f is ϵ -far from every function in \mathcal{F}_k , then Lemma 2 implies that $\text{NS}_\delta(f) > (k-1)\frac{\delta}{2}(1+\frac{\epsilon}{4})$, so that Lemma 3 implies that with probability at least $2/3$, $\hat{\text{NS}}_\delta(f) > (k-1)\frac{\delta}{2}(1+\frac{\epsilon}{4}) - 2\delta - \frac{\epsilon^3}{2^{10}} \geq (k-1)\frac{\delta}{2}(1+\frac{\epsilon}{8})$, and hence the tester will output REJECT. \square

Theorem 1 immediately follows from this result for any $k \geq 80/\epsilon$, since $m(\ell+1) \lesssim \frac{d}{\epsilon^{12}} \ln(\frac{1}{\epsilon})$ and $s \lesssim \frac{dk}{\epsilon^8} \ln(\frac{1}{\epsilon})$.

For $k < 80/\epsilon$, there is a trivial tester satisfying Theorem 1, based on the learn-then-validate trick of [3]. Specifically, in this case, we can take $s = \lceil \frac{c_1 dk}{\epsilon} \ln(2ek) \ln(\frac{1}{\epsilon}) \rceil + \lceil \frac{c_2}{\epsilon} \rceil \lesssim \frac{d}{\epsilon^2} \ln^2(\frac{1}{\epsilon})$ iid \mathcal{P} samples, for appropriate numerical constants $c_1, c_2 \geq 1$, and query for the f values for these s samples (so $q = s$ here). We then find a function $\hat{f} \in \mathcal{F}_k$ consistent with f on the first $\lceil \frac{c_1 dk}{\epsilon} \ln(2ek) \ln(\frac{1}{\epsilon}) \rceil$ of these (\hat{f} chosen independently from the rest of the samples), if such a function \hat{f} exists; we then check whether \hat{f} agrees with f on at least $(1-\epsilon/2) \lceil \frac{c_2}{\epsilon} \rceil$ of the remaining $\lceil \frac{c_2}{\epsilon} \rceil$ samples. If this \hat{f} exists and satisfies this condition, then we output ACCEPT, and otherwise we output REJECT. Since (one can show that) the graph dimension of \mathcal{F}_k is at most $4dk \log_2(2ek)$,² if $f \in \mathcal{F}_k$, then standard VC bounds for the realizable case [5, 6] imply that, for an appropriate choice of the numerical constant c_1 , with probability at least $5/6$, the function \hat{f} will have $\mathcal{P}(x : \hat{f}(x) \neq f(x)) < \epsilon/4$. Also, for an appropriately large numerical constant c_2 , a Chernoff bound implies that, with probability at least $5/6$, if $\mathcal{P}(x : \hat{f}(x) \neq f(x)) < \epsilon/4$ then \hat{f} will agree with f on at least $(1-\epsilon/2) \lceil \frac{c_2}{\epsilon} \rceil$ of the last $\lceil \frac{c_2}{\epsilon} \rceil$ samples. By the union bound, both of these events occur simultaneously with probability at least $2/3$, and the tester will output ACCEPT when they occur. On the other hand, if f is ϵ -far from every function in \mathcal{F}_k , then either \hat{f} will not exist (in which case the tester outputs REJECT), or else \hat{f} is some function in \mathcal{F}_k , so that $\mathcal{P}(x : \hat{f}(x) \neq f(x)) > \epsilon$. Therefore, for an appropriately large choice of the numerical constant c_2 , a Chernoff bound (applied under the conditional distribution given \hat{f}) and the law of total probability imply that, with probability at least $2/3$, if \hat{f} exists, then it disagrees with f on strictly more than $\frac{\epsilon}{2} \lceil \frac{c_2}{\epsilon} \rceil$ of the last $\lceil \frac{c_2}{\epsilon} \rceil$ samples, so that the tester will output REJECT.

2.3. Passive Testing

Next, we construct the passive tester achieving the bound in Theorem 2. Unlike the above general active tester, this construction follows more-closely the construction of a passive tester for unions of intervals from [2]. Take \mathcal{H} as the set of all constant functions $\mathcal{X} \rightarrow \mathcal{Y}$ (throughout this subsection), and

²This follows by considering the corresponding *shatter coefficient* values: that is, for any n distinct points $x_i \in \mathbb{R}$ and n values $y_i \in \mathcal{Y}$, the number of distinct $\{(x, g(x)) : x \in \mathcal{X}\} \cap \{(x_1, y_1), \dots, (x_n, y_n)\}$ sets that can be realized by functions $g \in \mathcal{F}_k$ is at most $(\frac{\epsilon n}{d})^{dk} (\frac{\epsilon n}{k})^k$ (obtained by applying Sauer's lemma within each region $(t_{j-1}, t_j]$, and multiplying by a bound on the effective number of possible placements of up to $k-1$ boundaries t_j between x_i points). This is strictly less than 2^n for any $n > 4dk \log_2(2ek)$.

let $\delta = \frac{\epsilon^2}{32k}$, $\gamma = \frac{\epsilon^3}{2^{11}}$, $m' = \left\lceil \frac{1}{2\gamma^2} \ln(12) \right\rceil$, $n = 1 + \left\lceil 2\sqrt{\lceil 1/\delta \rceil} \right\rceil$, and $s' = 2n \max\{2m', \lceil 8 \ln(6) \rceil\}$. The tester samples s' points $z'_1, \dots, z'_{s'}$ independent $\text{Uniform}(0, 1)$ and queries all of their f values. Let $t_1, \dots, t_{m'}$ be the first m' distinct values t in $\{1, \dots, s'/n\}$ for which $\exists i, j \in \{(t-1)n+1, \dots, tn\}$ with $i < j$ and $|z'_i - z'_j| < \delta$ while $z'_i \in (\delta, 1 - \delta)$, and for each $r \in \{1, \dots, m'\}$, denote by i_r the smallest integer $i > (t_r - 1)n$ with $\min_{j \in \{i+1, \dots, t_r n\}} |z'_i - z'_j| < \delta$ while $z'_i \in (\delta, 1 - \delta)$, and denote by j_r the smallest integer $j > i_r$ with $|z'_{i_r} - z'_{j_r}| < \delta$; if there do not exist m' such values t_r , then the tester may output any response, and this is considered a failure event. If these values do exist, then for each $r \leq m'$, denote $z_r = z'_{i_r}$ and $y_r = z'_{j_r}$. The tester then calculates the following quantity.

$$\hat{\text{NS}}'_\delta(f) = \frac{1 - 2\delta}{m'} \sum_{r=1}^{m'} \mathbb{I}[f(z_r) \neq f(y_r)].$$

We have the following lemma for this quantity.

Lemma 4. *For any measurable function $f : \mathcal{X} \rightarrow \mathcal{Y}$, with probability at least $2/3$, the values $t_1, \dots, t_{m'}$ exist, and $|\hat{\text{NS}}'_\delta(f) - \text{NS}_\delta(f)| \leq 2\delta + \gamma$.*

Proof. The existence of the values $t_1, \dots, t_{m'}$ (with high probability) follows from the so-called *birthday problem* as follows. Let $\beta = 1/\lceil 1/\delta \rceil$ and partition $[0, 1)$ into disjoint intervals $[(i-1)\beta, i\beta)$, $i \in \{1, \dots, 1/\beta\}$. For any n iid $\text{Uniform}(0, 1)$ samples w_1, \dots, w_n (for n as defined above), the probability none of these intervals contains more than one w_j value is $\prod_{j=1}^{n-1} (1 - j\beta) \leq \exp\{-\beta n(n-1)/2\}$, and noting that $n \geq 1 + 2\sqrt{1/\beta}$, this is at most e^{-2} . Furthermore, on the event that there exists at least one interval containing more than one w_j value, for \hat{j} defined as the first j such that $\exists j' < j$ with w_j and $w_{j'}$ in the same interval, we note that the (unconditional) distribution of $w_{\hat{j}}$ is $\text{Uniform}(0, 1)$. Therefore, with probability at least $1 - e^{-2} - 4\beta$, there exists some $i \in \{3, \dots, (1/\beta) - 2\}$ such that at least two w_j values are in $[(i-1)\beta, i\beta)$. Since $2\beta > 2\delta/(1+\delta) > \delta$, these intervals $[(i-1)\beta, i\beta)$ are strictly contained within $(\delta, 1 - \delta)$. Furthermore, since $\beta < \delta < 1/32$, we have $1 - e^{-2} - 4\beta > 1 - e^{-2} - 1/8 > 1/2$. Thus, for each $t \in \{1, \dots, s'/n\}$, the sequence $\{z'_i : i \in \{(t-1)n+1, \dots, tn\}\}$ has probability at least $1/2$ of containing a pair z'_i, z'_j ($i < j$) with $|z'_i - z'_j| < \delta$ and $z'_i \in (\delta, 1 - \delta)$. In particular, the expected number of indices t for which such a pair exists is at least $(1/2)s'/n \geq 2m'$. Since these sequences are independent (over t), a Chernoff bound implies that with probability at least $1 - \exp\{-(1/2)(s'/n)/8\} \geq 5/6$, at least m' of these sequences contain such a pair, so that on this event the values $t_1, \dots, t_{m'}$ indeed exist.

Next, note that since \mathcal{H} is the set of *constant* functions, for any x, x' and any $h \in \mathcal{H}_{(x, f(x))}$, we have $h(x') = h(x)$, which implies that for $x \sim \text{Uniform}(0, 1)$ and for $x' \sim \text{Uniform}(x - \delta, x + \delta)$ given x , we have $\text{NS}_\delta(f) = \mathbb{P}(f(x) \neq f(x'))$. Also note that, if they exist, the pairs (z_r, y_r) are iid, so that Hoeffding's in-

equality implies that with probability at least $1 - 2\exp\{-2\gamma^2 m'\} \geq 5/6$,

$$\left| \frac{1}{1-2\delta} \hat{\text{NS}}'_\delta(f) - \mathbb{P}(f(z_1) \neq f(y_1)) \right| \leq \gamma.$$

Furthermore, z_1 has distribution $\text{Uniform}(\delta, 1-\delta)$ and the conditional distribution of y_1 given z_1 is $\text{Uniform}(z_1 - \delta, z_1 + \delta)$. Therefore,

$$\begin{aligned} \mathbb{P}(f(z_1) \neq f(y_1)) &= \mathbb{P}(f(x) \neq f(x') | x \in (\delta, 1-\delta)) \\ &= \frac{1}{1-2\delta} \mathbb{P}(f(x) \neq f(x') \wedge x \in (\delta, 1-\delta)), \end{aligned}$$

and this rightmost quantity is at least as large as $\frac{1}{1-2\delta} (\mathbb{P}(f(x) \neq f(x')) - 2\delta) = \frac{1}{1-2\delta} (\text{NS}_\delta(f) - 2\delta)$ and at most as large as $\frac{1}{1-2\delta} \mathbb{P}(f(x) \neq f(x')) = \frac{1}{1-2\delta} \text{NS}_\delta(f)$. Altogether, on the above two events, we have

$$\begin{aligned} & \left| \hat{\text{NS}}'_\delta(f) - \text{NS}_\delta(f) \right| \\ & \leq |(1-2\delta)\mathbb{P}(f(z_1) \neq f(y_1)) - \text{NS}_\delta(f)| + \left| \hat{\text{NS}}'_\delta(f) - (1-2\delta)\mathbb{P}(f(z_1) \neq f(y_1)) \right| \\ & \leq 2\delta + \gamma(1-2\delta) \leq 2\delta + \gamma. \end{aligned}$$

To complete the proof, we note that both of these events occur simultaneously with probability at least $2/3$ by the union bound. \square

Finally, we have the following result on passive testing of piecewise constant functions.

Theorem 4. *If $k \geq 80/\epsilon$, then the procedure that outputs ACCEPT if $\hat{\text{NS}}'_\delta(f) \leq (k-1)\frac{\delta}{2}(1+\frac{\epsilon}{8})$, and otherwise outputs REJECT, is an s' -sample s' -query ϵ -tester for the class of k -piecewise constant functions under the distribution \mathcal{P} , for s' as defined above.*

Proof. This proof is nearly identical to that of Theorem 3. Recall from that proof that $2\delta + \gamma \leq (k-1)\frac{\delta}{2}\frac{\epsilon}{8}$. We are considering here the case of \mathcal{H} the set of constant functions.

If $f \in \mathcal{F}_k$, then Lemma 1 implies it has $\text{NS}_\delta(f) \leq (k-1)\frac{\delta}{2}$, so that Lemma 4 implies that with probability at least $2/3$, $\hat{\text{NS}}'_\delta(f) \leq (k-1)\frac{\delta}{2} + 2\delta + \gamma \leq (k-1)\frac{\delta}{2}(1+\frac{\epsilon}{8})$, and hence the tester will output ACCEPT.

On the other hand, if f is ϵ -far from every function in \mathcal{F}_k , then Lemma 2 implies that $\text{NS}_\delta(f) > (k-1)\frac{\delta}{2}(1+\frac{\epsilon}{4})$, so that Lemma 4 implies that with probability at least $2/3$, $\hat{\text{NS}}'_\delta(f) > (k-1)\frac{\delta}{2}(1+\frac{\epsilon}{4}) - 2\delta - \gamma \geq (k-1)\frac{\delta}{2}(1+\frac{\epsilon}{8})$, and hence the tester will output REJECT. \square

The first claim in Theorem 2 immediately follows from this theorem, noting that $s' \lesssim \frac{\sqrt{k}}{\epsilon^r} = O(\sqrt{k})$, and also noting that for any $k < 80/\epsilon$ we can obtain the result with the tester based on the learn-then-validate trick of [3],

as described above at the end of Section 2.2. In this latter case, the tester uses $\propto \frac{k}{\epsilon} \ln(2ek) \ln(\frac{1}{\epsilon}) \lesssim \frac{1}{\epsilon^2} \ln^2(\frac{1}{\epsilon})$ samples (since $d = 1$ when \mathcal{H} is the set of constant functions).

For the second claim in Theorem 2, first note that our assumption of $d > 0$ implies, in the case of \mathcal{H} the set of constant functions, that $|\mathcal{Y}| \geq 2$. Therefore, we can *reduce* testing unions of $\lfloor (k-1)/2 \rfloor$ intervals to testing k -piecewise constant functions by associating the binary labels 0 and 1 with any two distinct labels $y_0, y_1 \in \mathcal{Y}$. Then any $\{0, 1\}$ -valued function f_{01} can be mapped to a corresponding $\{y_0, y_1\}$ -valued function f , where f is a k -piecewise constant function if f_{01} is a union of $\lfloor (k-1)/2 \rfloor$ intervals, while f is ϵ -far from any k -piecewise constant function if f_{01} is ϵ -far from any union of $\lfloor (k-1)/2 \rfloor + 1$ intervals. This increase by one in the latter case is because the complement of a union of $\lfloor (k-1)/2 \rfloor$ intervals is also k -piecewise constant, but is possibly only representable as a union of $\lfloor (k-1)/2 \rfloor + 1$ intervals. To account for this increase by one, noting that the claim of an $\Omega(\sqrt{k})$ lower bound only regards large values of k , if we suppose $k > 1/\epsilon$, then any union of $\lfloor (k-1)/2 \rfloor + 1$ intervals is within distance $1/k < \epsilon$ of a union of $\lfloor (k-1)/2 \rfloor$ intervals. Thus, f_{01} is ϵ -far from any union of $\lfloor (k-1)/2 \rfloor + 1$ intervals if it is 2ϵ -far from any union of $\lfloor (k-1)/2 \rfloor$ intervals. Altogether, we have that f is ϵ -far from any k -piecewise constant function if f_{01} is 2ϵ -far from any union of $\lfloor (k-1)/2 \rfloor$ intervals. So, by this reduction, the $\Omega(\sqrt{\lfloor (k-1)/2 \rfloor}) = \Omega(\sqrt{k})$ lower bound of [3, 2] for passive testing of unions of $\lfloor (k-1)/2 \rfloor$ intervals implies a corresponding $\Omega(\sqrt{k})$ lower bound for testing k -piecewise constant functions.³ This completes the proof of Theorem 2.

3. Open Problem on the Query Complexity of Testing Polynomials

While the result above for active testing, when specialized to testing k -piecewise degree- p polynomials, obtains the optimal dependence on the number of pieces k , we were not able to show optimality in the degree p . This leads to an even more-basic question:

Open Problem: What is the optimal dependence on p in the query complexity of testing degree- p real-valued polynomials under $\mathcal{P} = \text{Uniform}(0, 1)$?

This question is open at this time, for both the active and passive property testing settings, and indeed also for the stronger value-query setting (where the

³Technically, the result of [3] establishes this lower bound for the problem of testing whether f_{01} is a union of n intervals or returns uniform random $\{0, 1\}$ labels. However, essentially the same argument would apply if, in the latter case, instead of random labels, we took f_{01} to be a randomly-chosen binary *function* based on a partition of $[0, 1]$ into $n' \gg n$ equal-sized regions, which would be at least $1/4 > 2\epsilon$ distance from any union of n intervals with very high probability.

tester can query for $f(x)$ at any $x \in \mathcal{X}$). There is a trivial $O(p)$ upper bound for both active and passive testing, based on the learn-then-validate trick of [3], since the graph dimension is $O(p)$. One might naïvely think that, since it is possible to fit any $p + 1$ values (at distinct x 's) with a degree- p polynomial, a *lower bound* of $\Omega(p)$ should also hold. However, we note that it is also possible to fit any k values (at distinct x 's) with a k -piecewise constant function, and yet above we proved that $O(1)$ active queries or $O(\sqrt{k})$ passive queries suffice to test k -piecewise constant functions. So the mere ability to fit $p + 1$ arbitrary values with a degree- p polynomial is not in-itself sufficient as a basis for proving a lower bound on the query complexity of testing degree- p polynomials.

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